14 Partial Derivatives

14.1 Functions of Several Variables

1. A function f of two variables is a function that assigns to each order pair (x, y) of the domain, a unique order pair $f(x, y)$ in the range (it maps a region from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$).

2. if we let $z = f(x, y)$, then the variables x and y are the independent variables (the input), and the variable z is the dependent one (output)

3. the domain of f: the values (x, y) for which f is defined (i.e. the values that can be plugged into the function)

4. the graph of f is the set of points $(x, y, f(x, y))$ in $\mathbb{R}^3$

5. a linear function is a function whose terms are linear: $f(x, y) = ax + by + c$, where a, b, c are constants. Its graph is the plane $ax + by - z + c = 0$

6. the level curves of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant chosen from the range of f (i.e. the graph at the height $z = k$). Closer the level curves are, steeper the surface is (and so the surface is flatter where the level curves are farther apart).

7. examples: topographic maps of mountainous regions (where each level curve describes the shape at a particular height) or isothermals (regions with the same temperature are in between the level curves).

8. for three or more variables, similar results hold, where f maps an ordered n-tuple (like the 3-tuple (x, y, z)) to the real number value $f((x_1, x_2, \ldots x_n))$

14.2 Limits and Continuity

1. $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $f(x,y) \to L$ and $(x,y) \to (a,b)$

2. first try to plug in the values a and b into $f(x,y)$ to get a definite answer.

3. if that doesn't work, try to plug in the values a and b into $f(x,y)$ along different paths; e.g. if $(x,y) \to (0,0)$ approach the point $(0,0)$ first along the x-axis by letting $y = 0$ (and then find $\lim_{x\to 0} f(x,0)$ from the negative and positive side), and then along the y-axis by letting $x = 0$ (and then find $\lim_{y\to 0} f(0,y)$ from the negative and positive side). If the two limits exist and they are equal, then the limit of $f(x,y)$ exists as well.

4. if $f(x,y,z)$ is of three variables: (1) let $x = 0, y = 0$, and take $\lim_{z\to 0} f(0,0,z)$ (2) let $x = 0, z = 0$, and take $\lim_{y\to 0} f(0,0,y)$, and (3) let $y = 0, z = 0$, and take $\lim_{x\to 0} f(x,0,0)$. If all three limits exist and they are equal, then the original limit exists. Otherwise, limit DNE.

5. a function $f(x,y)$ of two variables is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

6. a function is continuous on a set if it is continuous at every point of the set

7. polynomials are continuous on their domain: $\mathbb{R}^2$

8. ratio of polynomials is continuous on its domain: $\mathbb{R}^2$ except the roots of the denominator

9. for piecewise functions one should always check continuity at the the points where it pieces together

10. composition, multiplication, addition and subtraction of continuous functions is continuous (be careful with division –see item 8 above)

14.3 Partial Derivatives

1. partial derivative with respect to x of $f(x,y)$ at (a,b) is the change of f in the x-direction

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

   (similarly for $f_y(a,b)$)

2. partial derivative with respect to x of $f(x,y)$ (regard y as a constant) is

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

   and it is the slope of the tangent line to the trace of the curve in the plane $y =$constant

3. partial derivative with respect to y of $f(x,y)$ (regard x as a constant) is the change of f in the y-direction

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

   and it is the slope of the tangent line to the trace of the curve in the plane $x =$constant

4. the above definitions generalize if the curve $u = f(x_1, x_2, \ldots, x_n)$ has n variables.

5. second partial derivative = the 2nd derivative with the same variable or a different variable

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = (f_x)_x$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = (f_x)_y$$

   (the first two symbols in each string are the preferred ones)

6. note that generally $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \neq \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$. However if both partials are continuous on the domain, then they are equal

7. skip partial differential equations and the Cobb-Douglas Production function.

14.4 Tangent Planes and Linear Approximations

Tangent Planes

1. let S be a surface containing the point $P(x_0, y_0, z_0)$, and let C_1 and C_2 be the curves obtained by intersecting S and the planes $y =$ constant y_0 and $x =$ constant x_0, respectively. The tangent plane to the surface S at P is the plane that contains both tangent line T_1 and T_2 to C_1 and C_2, respectively. This is the plane that approximates S best near the point P

2. equation of the tangent plane:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

   where (A, B, C) is the normal to the tangent plane, and $P(x_0, y_0, z_0)$ is the point of intersection of the tangent plane and the surface S

3. the above equation can also be found using the equation:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

   where $z = f(x, y)$

Linear Approximations

1. approximating the equation of a surface with a linear equation near a point

2. the tangent plane equation is a good linear approximation at the point P, and it can be used the approximate the function at nearby points.

3. for continuous partial derivatives of f, we define the linearization of the function $f(x, y)$ at (a, b) is

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b),$$

   and the linear approximation is

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b),$$

4. let $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

   where Δx and Δy are small changes in x and y

5. convenient conditions for differentiability: If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b)

Differentials

1. for a differentiable function of one variable $y = f(x)$, we define the differential dx to be an independent variable that shows a small change in x. Then $dy = f'(x)dx$ is the differential of y, and it represents the change in the tangent line as we changed x by $\Delta x = dx$ (note that $\Delta y = f(x + dx) - f(x)$ is the change in the function as we changed x by $\Delta x = dx$).

2. for a differentiable function of two variables $z = f(x, y)$, we define the differentials dx and dy to be the independent variable change (so they can be given any small values), and then we can find the differential $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Functions of Three or More Variables

1. the linear approximation is

$$f(x, y, z) \approx f(a, b, c) + \frac{\partial f}{\partial x}(a, b, c)(x - a) + \frac{\partial f}{\partial z}(a, b, c)(z - c)$$

2. the linearization is

$$L(x, y, z) = f(a, b, c) + \frac{\partial f}{\partial x}(a, b, c)(x - a) + \frac{\partial f}{\partial z}(a, b, c)(z - c)$$

3. the differential of the function $w = f(x, y, z)$ is

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial w}dw$$

14.5 Chain Rule

1. if $z = f\Big(x(t), y(t)\Big)$, then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. if $z = f\Big(x(s,t), y(s,t)\Big)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

3. in the equations above, we have independent variables s and t, intermediate variables x and y, and dependent variable z

4. the above equations can be generalized for functions of n variables, each of which can be a function of m variables (page 934)

5. Implicit Differentiation helps find derivatives of $y(x)$ using an equation $F(x,y) = 0$ (that is not necessarily in a nice and easy form to work with, i.e. find $\frac{dy}{dx}$ for the using the equation $x^3 - 2y^5 + 3xy^2 = 13$) by solving for $\frac{dy}{dx}$ in the equation $F'(x,y) = 0$.

6. If $F(x,y) = 0$ is defined on a disk containing the point (a,b) where $F(a,b) = 0$ and $\frac{\partial F}{\partial y} \neq 0$ and both partials are continuous on the disk, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

7. If $F(x,y,z) = 0$ is defined within a sphere containing the point (a,b,c) and $\frac{\partial F}{\partial z} \neq 0$ and all three partials are continuous, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

14.6 Directional Derivative and the Gradient Vector

1. the directional derivative enables us to find the rate of change of a function of two or more variables

2. recall that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ give the rate of change of f in the x- and y-direction (i.e. in the direction of the unit vectors $i =< 1, 0 >$ and $j =< 0, 1 >$). The rate of change of $z = f(x, y)$ in the direction of an arbitrary **unit** vector $u =< a, b >$ (make sure that u is a unit vector) is

$$D_u f(x, y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

   or

$$D_u f(x, y) =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \cdot < a, b >$$

3. The rate of change of $f(x, y, z)$ in the direction of an arbitrary **unit** vector $u =< a, b, c >$ (make sure that u is a unit vector) is

$$D_u f(x, y, z) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b + \frac{\partial f}{\partial z} c$$

4. the gradient vector, del f, is the vector

$$\textbf{grad } f = \nabla f =< \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} >$$

   or for three variables we have

$$\textbf{grad } f = \nabla f =< \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z} >$$

5. and so

$$D_u f(x, y) = \nabla f \cdot u$$

6. the maximum of the directional derivative $D_u f$ of a differentiable function f is $|\nabla f|$ and it occurs at a vector $\mathbf{u}$ that has the same direction as the gradient ∇f (that is so because we want to maximize $D_u = \nabla f \cdot u = |\nabla f||u| \cos\theta$)

tangent planes to level surfaces

1. Let S be a level surface that has the equation $F(x, y, z) = k$, $P(x_0, y_0, z_0)$ a point on S, and C a curve on S through P given by the equation $r(t) =< x(t), y(t), z(t) >$. Then the gradient at P is perpendicular to the vector $r'(t)$ that is tangent to the curve C. That is to say that ∇F is the normal of the plane that contains all the tangent lines at P, i.e. ∇F is the normal of the tangent plane to the surface S. The equation of this plane is

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

2. and so the normal vector at P of the plane at is given by the gradient vector $\nabla F(x_0, y_0, z_0)$

3. and also the normal line to S at P has the symmetric equations

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

significance of the Gradient Vector

1. the gradient is orthogonal to the level surface S of f at P

2. it gives the direction of the fastest increase of a function

3. on contour maps it points “uphill” and perpendicular to the level curves